REFLEXIVITY OF ISOMETRIES OF ORDER N

ABDULLAH BIN ABU BAKER

Dedicated to the memory of Professor Sudipta Dutta

Abstract. In this paper, we prove that if the group of isometries on $C_0(\Omega, X)$ is algebraically reflexive, then the set of isometries of order n on $C_0(\Omega, X)$ is also algebraically reflexive. Here, Ω is a first countable locally compact Hausdorff space, and X is a Banach space having the strong Banach-Stone property. As a corollary to this, we establish the algebraic reflexivity of the set of generalized bi-circular projections on $C_0(\Omega, X)$.

1. Introduction

Reflexivity and hyperreflexivity explores the relation between sets of operators and their common invariant subspaces. The notion of reflexivity was introduced by Halmos for lattices of closed subspaces of a Hilbert space $\mathcal{H}$, [10]. If $\mathcal{A}$ is a subset of $B(\mathcal{H})$, then Lat $\mathcal{A}$ denotes the set of all subspaces invariant under every operator in $\mathcal{A}$. If $\mathcal{L}$ is a collection of closed subspaces of $\mathcal{H}$, then Alg $\mathcal{L}$ denotes the algebra of all operators which leave every subspace in $\mathcal{L}$ invariant. A lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = \text{Lat Alg } \mathcal{L}$. Reflexivity for algebras was introduced by Radjavi and Rosenthal, [20]. An algebra $\mathcal{A}$ is called reflexive if $\mathcal{A} = \text{Alg Lat } \mathcal{A}$.

Loginov and Šul′man, [17], extended this notion to linear subspaces of $B(\mathcal{H})$. For any subspace $\mathcal{S}$ of $B(\mathcal{H})$, define

$$\text{Ref } \mathcal{S} = \{T \in B(\mathcal{H}) : Th \in \overline{\mathcal{S}h}, \ \forall \ h \in \mathcal{H}\}. \tag{1.1}$$

Ref $\mathcal{S}$ is called the attached space for $\mathcal{S}$ or the topological closure of $\mathcal{S}$. The subspace $\mathcal{S}$ is called reflexive (or topologically reflexive) if $\mathcal{S} = \text{Ref } \mathcal{S}$.

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Throughout this paper, let X be a Banach space and $B(X)$ be the algebra of all bounded linear operators on X. One can see that in definition (1.1), the assumptions that the underlying space is a Hilbert space and $\mathcal{S}$ is a linear subspace are not essential. We can define the topological closure for arbitrary subset $\mathcal{S}$ of $B(X)$.

If $\mathcal{A}$ is an algebra that contains the identity, then Lat $\mathcal{A}$ is determined by the closed cyclic subspaces of $\mathcal{A}$, so Alg Lat $\mathcal{A}$ = Ref $\mathcal{A}$. Thus, both definitions coincides for unital algebras.

The notion of algebraic reflexivity has appeared in different contexts. The term was first coined by Hadwin [7]. Let V be a vector space over a field $\mathbb{F}$, and let $\mathcal{L}(V)$ denotes the algebra of all linear transformations on V. For a subspace $\mathcal{S}$ of $\mathcal{L}(V)$ define

$$\overline{\mathcal{S}}^a = \{T \in \mathcal{L}(V) : Tx \in \mathcal{S}x,\ \forall\ x \in V\}.$$

So, $T \in \overline{\mathcal{S}}^a$ if and only if for each $x \in V$, $\exists\ S \in \mathcal{S}$, depending on x, such that $Tx = Sx$. We say that T interpolates $\mathcal{S}$ or T is locally in $\mathcal{S}$. Obviously, $\mathcal{S} \subseteq \overline{\mathcal{S}}^a$. The subspace $\mathcal{S}$ is called algebraically reflexive if $\mathcal{S} = \overline{\mathcal{S}}^a$.

Obviously, any topologically reflexive subspace is algebraically reflexive.

Algebraic reflexivity in general and on certain classes of transformations were studied by many authors, see for instance, [4, 5, 7, 14, 16, 19, 21] and [22]. Lecture Notes by Molnár [18] gives a very comprehensive account of this theory.

An important class of transformations in $B(X)$ is the group of surjective linear isometries, denoted by $\mathcal{G}(X)$. We denote by $\mathcal{G}^n(X)$, all operators T in $\mathcal{G}(X)$ such that $T^n = I$. Such operators are called isometries of order n. An operator $T \in \overline{\mathcal{G}(X)}^a$ is called a local isometry.

The isometry group of any finite dimensional Banach space is algebraically reflexive. Every Banach space admits a renorming whose isometry group is algebraically reflexive, [13]. The isometry group of any infinite dimensional Hilbert space fails to be algebraically reflexive. Indeed, given $x, y \in \mathcal{H}$ such that $||x|| = ||y||$, there exists $T \in \mathcal{G}(\mathcal{H})$ such that $T(x) = y$. So, $\overline{\mathcal{G}(\mathcal{H})}^a$ contains all into isometries.

For a little nontrivial, but an easy, example one can look at ℓ_∞ in $B(\ell_2)$. We know that $B(\ell_2)$ contains an isometric copy of ℓ_∞. We will see later that ℓ_∞ is algebraically reflexive.

In [5], Dutta and Rao proved that for a compact Hausdorff space Ω, if $\mathcal{G}(C(\Omega))$ is algebraically reflexive, then $\mathcal{G}^2(C(\Omega))$ is also algebraically reflexive. Motivated by this result, in this paper we investigate the algebraic reflexivity of isometries of order n on $C_0(\Omega, X)$, the space of X-valued continuous functions on a first countable locally compact Hausdorff space Ω vanishing at infinity.

A projection $P \in B(X)$ is said to be a generalized bi-circular projection if $P + \lambda(I - P) \in \mathcal{G}(X)$, where λ is a unit modulus complex number not equal to 1. This class was introduced by Fošner, Iliševič and Li [6] in 2007. Description of generalized bi-circular projections for different Banach spaces can be found in [1, 3, 12] and [15]. As a corollary to our result, we establish the algebraic reflexivity of the set of generalized bi-circular projections on $C_0(\Omega, X)$, which answers a question raised in by Dutta and Rao in [5].

2. Preliminaries

The study of isometries between Banach spaces is one of the most important research areas in functional analysis. One of the most classical results in this area is the Banach-Stone theorem describing surjective linear isometries between Banach spaces of complex-valued continuous functions on compact Hausdorff spaces.

While investigating reflexivity problems of the isometry group of a Banach space X, firstly we observe that any local isometry on X is actually an isometry. In particular, if $T \in \overline{\mathcal{G}(X)}^a$, then for any $x \in X$, there exists $T_x \in \mathcal{G}(X)$ such that $T(x) = T_x(x)$. Now, taking norm on both sides we get $\|T(x)\| = \|T_x(x)\| = \|x\|$. So, in order to show that $\mathcal{G}(X)$ is algebraically reflexive, we need to prove that any local isometry is surjective. But this problem is not as simple as it seems. Secondly, since we have a precise description of surjective isometries for most of the classical Banach spaces, one has a good idea how any local isometry looks like.

For the sake of completeness we recall the Banach-Stone theorem and some other definitions from [11, Chapter I] which are needed for the vector-valued version.

Theorem 2.1. [2, Theorem 7.1] *Let Ω be a locally compact Hausdorff space. If $T : C_0(\Omega) \longrightarrow C_0(\Omega)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$, and a continuous map $u : \Omega \longrightarrow \mathbb{T}$ such that*

$$Tf(\omega) = u(\omega)f(\phi(\omega)), \quad \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

Here, $\mathbb{T}$ denotes the unit circle in the complex plane.

Definition 2.2. Let $T \in B(X)$.

(1) The operator T is called a multiplier of X if for every element $p \in ext(B_{X^*})$, there exists $a_T(p) \in \mathbb{C}$ such that $T^*p = a_T(p)p$. The collection of all multipliers is denoted by $Mult(X)$. Here, $ext(B_{X^*})$ denotes the set of extreme points of B_{X^*}

(2) The centralizer of X is defined as

$Z(X) = \{T \in Mult(X) : \exists\ \overline{T} \in Mult(X)$ such that $a_{\overline{T}}(p) = \overline{a_T(p)},\ \forall\ p \in ext(B_{X^*})\}$.

Definition 2.3. A Banach space X is said to have trivial centralizer if the dimension of $Z(X)$ is equal to 1; that is, if the only elements in the centralizer are scalar multiples of the identity operator I. Obviously, this is true if X is itself the scalar field.

Theorem 2.4. [2, Theorem 8.10] *Let Ω be a locally compact Hausdorff space, and let X be a Banach space with trivial centralizer. If $T : C_0(\Omega, X) \longrightarrow C_0(\Omega, X)$ is a surjective isometry, then there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow \mathcal{G}(X)$, continuous with respect to the strong operator topology of $B(X)$, such that*

$$Tf(\omega) = u_\omega(f(\phi(\omega))),\ \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

For simplicity, we denote $u(\omega)$ by u_ω.

Definition 2.5. [2, Definition 8.2] A Banach space X is said to have the strong Banach-Stone property if it satisfies the condition in Theorem 2.4.

It is known that strictly convex spaces have trivial centralizer. In particular, they have the strong Banach-Stone property.

Before we proceed let us see how ℓ_∞ is algebraically reflexive in $B(\ell_2)$.

Let $T \in \overline{\ell_\infty}^a$. Then, for each $f \in \ell_2$ we have $Tf(j) = \phi_f(j)f(j)$ for some $\phi_f \in \ell_\infty$. Hence, for the standard unit vectors e_j in ℓ_2,

$$Te_j(k) = \phi_{e_j}(k)e_j(k) = \begin{cases} \phi_{e_j}(j), & \text{for } j = k \\ 0, & \text{for } j \neq k. \end{cases}$$

This implies that $Te_j = \phi_{e_j}(j)e_j$. Now, for $f \in \ell_2$ we have

$$Tf = T\left(\sum_{j=1}^{\infty} f(j)e_j\right) = \sum_{j=1}^{\infty} f(j)\phi_{e_j}(j)e_j = f\phi,$$

where, $\phi = (\phi_{e_j}(j))$. As T is a bounded linear operator, $\phi \in \ell_\infty$. Therefore, $T \in \ell_\infty$.

We can actually show that ℓ_∞ is topologically reflexive.

The following lemma will be useful later.

Lemma 2.6. *Let $T \in \mathcal{G}(C_0(\Omega, X))$. Then T is an isometry of order n if and only if there exist a homeomorphism ϕ of Ω and a map $u : \Omega \longrightarrow \mathcal{G}(X)$ satisfying*

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I, \quad \phi^n(\omega) = \omega, \quad \forall\, \omega \in \Omega;$$

where I denotes the identity map on X and T is given by

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\, f \in C_0(\Omega),\ \omega \in \Omega.$$

Proof. Since $T \in \mathcal{G}(C_0(\Omega, X))$, $\exists$ a homeomorphism $\phi : \Omega \longrightarrow \Omega$ and a map $u : \Omega \longrightarrow \mathcal{G}(X)$ such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))), \quad \forall\, f \in C_0(\Omega),\ \omega \in \Omega.$$

As $T \in \mathcal{G}^n(C_0(\Omega, X))$ we have $T^n f(\omega) = f(\omega)$. This show that

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)}(f(\phi^n(\omega))) = f(\omega). \tag{2.1}$$

For a fixed $x \in X$ and $\omega \in \Omega$ consider the function $f_x \in C_0(\Omega, X)$ such that $f_x(\omega) = x$. Applying Equation (2.1) to f_x we get

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)}(x) = x.$$

Since this can be done for each $x \in X$ and each $\omega \in \Omega$ we conclude

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I.$$

This also implies that $f(\phi^n(\omega)) = f(\omega)$ for all $f \in C_0(\Omega, X)$. Hence, $\phi^n(\omega) = \omega$. $\square$

3. Algebraic Reflexivity of $\mathcal{G}^n(C_0(\Omega, X))$

Our main result is the following.

Theorem 3.1. *Let Ω be a first countable locally compact Hausdorff space, and let X be a Banach space which has the strong Banach-Stone property. If $\mathcal{G}(C_0(\Omega, X))$ is algebraically reflexive, then $\mathcal{G}^n(C_0(\Omega, X))$ is algebraically reflexive.*

Proof. Let $T \in \overline{\mathcal{G}^n(C_0(\Omega, X))}^a$. Then for each $f \in C_0(\Omega, X)$ we have $Tf(\omega) = u^f_\omega(f(\phi_f(\omega)))$ where $u^f : \Omega \longrightarrow \mathcal{G}(X)$ is continuous in strong operator topology and satisfies

$$u^f_\omega \circ u^f_{\phi_f(\omega)} \circ \cdots \circ u^f_{\phi_f^{n-1}(\omega)} = I,$$

and ϕ_f is a homeomorphism of Ω such that $\phi^n_f(\omega) = \omega$ for all $\omega \in \Omega$. In particular $T \in \overline{\mathcal{G}(C_0(\Omega, X))}^a$. Hence, there exist a homeomorphism $\phi : \Omega \longrightarrow \Omega$, and a map $u : \Omega \longrightarrow \mathcal{G}(X)$ such that

$$Tf(\omega) = u_\omega(f(\phi(\omega))),\ \forall\ f \in C_0(\Omega),\ \omega \in \Omega.$$

To show that $T \in \mathcal{G}^n(C_0(\Omega, X))$, we need to prove that $T^n = I$, that is, by Lemma 2.6

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I \text{ and } \phi^n(\omega) = \omega,\ \forall\ \omega \in \Omega.$$

Suppose $f = h \otimes x$, where h is a strictly positive function in $C_0(\Omega)$ and $0 \neq x \in X$. Then we have $Tf(\omega) = u^f_\omega(f(\phi_f(\omega))) = u_\omega(f(\phi(\omega)))$ or $u^f_\omega(h(\phi_f(\omega))x) = u_\omega(h(\phi(\omega))x)$. Taking norm on both sides ans using the fact that $u^f_\omega,\ u_\omega$ are isometries, and h is strictly positive we get $u^f_\omega(x) = u_\omega(x)$. Therefore, $u^f_\omega = u_\omega$ for all $\omega \in \Omega$.

Let ω be any point in Ω. We consider the following cases.

Case I. Assume that $\omega = \phi(\omega)$. Then

$$\phi^n(\omega) = \phi(\phi(\cdots(\phi(\omega))\cdots))\ (n \text{ times}) = \omega.$$

We choose $h \in C_0(\Omega)$ such that $0 < h(\omega) \leq 1$ and $h^{-1}\{1\} = \{\omega\}$. For $f = h \otimes x$, $0 \neq x \in X$, evaluating Tf at ω we get

$$\begin{aligned}
Tf(\omega) &= u_\omega(f(\phi(\omega))) = u_\omega^f(f(\phi_f(\omega))) \\
&\Longrightarrow u_\omega(h(\phi(\omega))x) = u_\omega^f(h(\phi_f(\omega))x) \\
&\Longrightarrow u_\omega(x) = u_\omega^f(h(\phi_f(\omega))x) \quad (\because\ h(\phi(\omega)) = h(\omega) = 1) \\
&\Longrightarrow \|u_\omega(x)\| = \|u_\omega^f(h(\phi_f(\omega))x)\| \\
&\Longrightarrow h(\phi_f(\omega)) = 1 \quad (u_\omega \text{ and } u_\omega^f \text{ are isometries}) \\
&\Longrightarrow \phi_f(\omega) = \omega \quad (\text{by the choice of } h) \\
&\Longrightarrow \phi_f^2(\omega) = \cdots = \phi_f^{n-1}(\omega) = \omega.
\end{aligned}$$

So, we have

$$\begin{aligned}
I &= u_\omega^f \circ u_{\phi_f(\omega)}^f \circ \cdots \circ u_{\phi_f^{n-1}(\omega)}^f \\
&= u_\omega^f \circ u_\omega^f \circ \cdots \circ u_\omega^f \\
&= u_\omega \circ u_\omega \circ \cdots \circ u_\omega \text{ (as } u_\omega^f = u_\omega).
\end{aligned}$$

Case II. We assume that $\phi(\omega) \neq \omega$, $\phi^m(\omega) = \omega$ such that m divides n and $\phi^s(\omega) \neq \omega$ for all $s < m$.

As m divides n, there exist some positive integer q such that $n = mq$. Therefore, we have

$$\phi^n(\omega) = \phi^{mq}(\omega) = \phi^m(\phi^m(\cdots(\phi^m(\omega)))\cdots) \text{ (q times)} = \omega.$$

We now choose $h \in C_0(\Omega)$ such that $1 \leq h(\omega) \leq m$ and

$$h^{-1}\{1\} = \{\omega\},\ h^{-1}\{2\} = \{\phi(\omega)\}, \ldots,\ h^{-1}\{m\} = \{\phi^{m-1}(\omega)\}.$$

Let $f = h \otimes x$ for $0 \neq x \in X$. Evaluating Tf at $\omega, \phi(\omega), \ldots, \phi^{m-1}(\omega)$ and considering our choice of the function h we get $\phi_f^p(\omega) = \phi^p(\omega)$ for $1 \leq p \leq m$.

This implies that

$$\phi_f^{m+1}(\omega) = \phi_f(\phi_f^m(\omega)) = \phi_f(\omega) = \phi(\omega) = \phi(\phi^m(\omega)) = \phi^{m+1}(\omega).$$

Thus, $\phi_f^p(\omega) = \phi^p(\omega),$ for $m+1 \leq p \leq n-1$. Since $u_\omega = u_\omega^f$ for all $\omega \in \Omega$, we have

$$u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = u_\omega^f \circ u^f_{\phi_f(\omega)} \circ \cdots \circ u^f_{\phi_f^{n-1}(\omega)} = I.$$

Case III. We assume that $\phi(\omega) \neq \omega$, $\phi^m(\omega) = \omega$ such that m does not divides n and $\phi^s(\omega) \neq \omega$ for all $s < m$.

There exist integers r and q such that $n = mq + r$, $0 < r < m$. We choose $h \in C_0(\Omega)$ such that $1 \leq h(\omega) \leq m$ and

$$h^{-1}\{1\} = \{\omega\},\ h^{-1}\{2\} = \{\phi(\omega)\}, \ldots,\ h^{-1}\{m\} = \{\phi^{m-1}(\omega)\}.$$

By applying Tf at $\omega,\ \phi(\omega), \ldots,\ \phi^{m-1}(\omega)$ and proceeding in the same way as in **Case II** we get $\phi_f^p(\omega) = \phi^p(\omega)$ for $1 \leq p \leq n-1$. We now see that

$$\begin{aligned}
Tf(\phi^{n-1}(\omega)) \quad &= \quad u_{\phi^{n-1}(\omega)}(f(\phi^n(\omega))) = u^f_{\phi^{n-1}(\omega)}(f(\phi_f(\phi^{n-1}(\omega)))) \\
&\implies \quad u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\phi_f(\phi_f^{n-1}(\omega)))x) \\
&\implies \quad u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\phi_f^n(\omega))x) \\
&\implies \quad u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(h(\omega)x) \quad (\because\ \phi_f^n(\omega) = \omega) \\
&\implies \quad u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x) = u^f_{\phi^{n-1}(\omega)}(x) \quad (\because\ h(\omega) = 1) \\
&\implies \quad \|u_{\phi^{n-1}(\omega)}(h(\phi^n(\omega))x)\| = \|u^f_{\phi^{n-1}(\omega)}(x)\| \\
&\implies \quad h(\phi^n(\omega)) = 1 \quad (u_{\phi^{n-1}(\omega)} \text{ and } u^f_{\phi^{n-1}(\omega)} \text{ are isometries}) \\
&\implies \quad \phi^n(\omega) = \omega \quad (\text{by the choice of } h).
\end{aligned}$$

But, our assumption that $\phi^m(\omega) = \omega$ implies that $\phi^{mq}(\omega) = \omega$. Hence, we have

$$\omega = \phi^n(\omega) = \phi^{r+mq}(\omega) = \phi^r(\phi^{mq}(\omega)) = \phi^r(\omega),$$

a contradiction because $r < m$.

Case IV. We assume that $\omega,\ \phi(\omega), \ldots,\ \phi^{n-1}(\omega)$ are all distinct.

Choose $h \in C_0(\Omega)$ such that $1 \leq h(\omega) \leq n$ and

$$h^{-1}\{1\} = \{\omega\},\ h^{-1}\{2\} = \{\phi(\omega)\}, \ldots,\ h^{-1}\{n\} = \{\phi^{n-1}(\omega)\}.$$

Proceeding the same way as in **Case III** we get

$$\phi^n(\omega) = \omega \text{ and } u_\omega \circ u_{\phi(\omega)} \circ \cdots \circ u_{\phi^{n-1}(\omega)} = I.$$

This completes the proof. □

Corollary 3.2. *Let Ω be a first countable locally compact Hausdorff space. Let X be a Banach space which has the strong Banach-Stone property, and does not have any generalized bi-circular projections. If $\mathcal{G}(C_0(\Omega, X))$ is algebraically reflexive, then the set of generalized bi-circular projections on $C_0(\Omega, X)$ is also algebraically reflexive.*

Proof. We denote the set of all generalized bi-circular projections on $C_0(\Omega, X)$ by $\mathcal{P}$. Let $P \in \overline{\mathcal{P}}^a$. Then for each $f \in C_0(\Omega, X)$, there exist $P_f \in \mathcal{P}$ such that $Pf = P_f f$. Therefore, by [1, Theorem 4.2] and the assumption on X, for each f there exists a homeomorphism ϕ_f of Ω, $u^f : \Omega \longrightarrow \mathcal{G}(X)$ satisfying

$$\phi_f^2(\omega) = \omega \text{ and } u_\omega^f \circ u_{\phi_f(\omega)}^f = I, \quad \forall\ \omega\ \in\ \Omega$$

such that

$$Pf(\omega) = \frac{1}{2}[f(\omega) + u_\omega^f(f(\phi_f(\omega)))].$$

Therefore, for each $f \in C_0(\Omega, X)$, we get $(2P - I)f(\omega) = u_\omega^f(f(\phi_f(\omega)))$. This implies that $2P - I \in \overline{\mathcal{G}^2(C_0(\Omega, X))}^a$. The conclusion follows from Theorem 3.1. □

Combining Theorem 3.1 with [14, Theorem 7] we have the following corollary.

Corollary 3.3. *Let Ω be a first countable compact Hausdorff space, and let X be a uniformly convex Banach space such that $\mathcal{G}(X)$ is algebraically reflexive. Then $\mathcal{G}^n(C(\Omega, X))$ is algebraically reflexive.*

4. Remarks

Until now we have not said anything about hyperreflexivity. The concept of hyperreflexivity is stronger than reflexivity. For a subspace $\mathcal{S}$ of $B(\mathcal{H})$, and $T \in B(\mathcal{H})$ we define

$$\alpha(T, \mathcal{S}) = \sup\{||P^\perp TQ|| : P, Q \text{ are projections and } P^\perp \mathcal{S} Q = \{0\}\}.$$

It is clear that $\alpha(T,\mathcal{S}) \leq \mathrm{dist}(T,\mathcal{S})$. The subspace $\mathcal{S}$ is said to be hyperreflexive if there is a constant K such that for every $T \in B(\mathcal{H})$ we have $\mathrm{dist}(T,\mathcal{S}) \leq K\alpha(T,\mathcal{S})$.

It is not very difficult to see that any hyperreflexive subspace is reflexive.

Hadwin, [8], introduced an asymptotic version of hyperreflexivity. Define the seminorm $d_a(\cdot,\mathcal{S})$ on $B(\mathcal{H})$ by

$d_a(T,\mathcal{S}) = \sup\{\limsup_\lambda ||P_\lambda T Q_\lambda|| : \{P_\lambda\}, \{Q_\lambda\}$ are nets of projections such that $||P_\lambda S Q_\lambda|| \longrightarrow 0 \ \forall\ S \in \mathcal{S}\}$.

The subspace $\mathcal{S}$ is said to be approximately hyperreflexive if there is a constant K such that $\mathrm{dist}(T,\mathcal{S}) \leq K d_a(T,\mathcal{S})$ for every $T \in B(\mathcal{H})$.

Hadwin proved that every unital C^*-algebra of $B(\mathcal{H})$ is approximately hyperreflexive, see [8, Theorem 13].

In [9], Hadwin defined an ingenious view of reflexivity in which he unified most of the versions of reflexivity as well as hyperreflexivity. He proved several results in this setting and obtained many important ones as corollaries to his results.

Yousefi, [23], used Hadwin's view of reflexivity to prove that every C^*-algebra $\mathcal{A}$ of $B(\mathcal{H})$ is approximately hyperreflexive, surprisingly, with an elementary method. We note that $\mathcal{A}$ does not have to be unital.

We want to emphasize here that Yousefi's approach to use Hadwin's general version is new and rather powerful and we think it could be explored further to study reflexivity and hyperreflexivity problems.

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Department of Applied Sciences, Indian Institute of Information Technology, Allahabad, India

E-mail address: abdullahmath@gmail.com